

On the operator content of nilpotent orbifold models

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1 Introduction

The present work is in essence a continuation of our paper [DM2]. To describe our results we assume that the reader is familiar with the theory of vertex operator algebras (VOA) and their representations (see for example [B], [FLM] and [FHL]).

Suppose that V is a *holomorphic* VOA and G is a finite (and faithful) group of automorphisms of V . It is then a general conjecture that the fixed vertex operator subalgebra V^G of G -invariants is *rational*, that is, V^G has a finite number of simple modules and every module for V^G is completely reducible. In fact, following the work of Dijkgraaf-Witten [DW] and Dijkgraaf-Pasquier-Roche [DPR], one can formulate a precise conjecture concerning the category $\text{Mod-}V^G$ of V^G -modules. Essentially it says that $\text{Mod-}V^G$ is equivalent to the category of $D^c(G)$ -modules, where $D^c(G)$ is the so-called quantum double of G [Dr] modified by a certain 3-cocycle $c \in H^3(G, S^1)$. This cocycle itself arises from a quasi-coassociative tensor product on $\text{Mod-}D^c(G)$ which is expected to reflect the algebraic properties of an appropriate notion of tensor product on the category $\text{Mod-}V^G$.

One of the goals of the present paper is to prove a variation on this theme for a broad class of finite groups G , not necessarily abelian, under a suitable hypothesis concerning the so-called *twisted sectors* for V . Let us explain these results in more detail.

For each $g \in G$ we have the notion of a g -twisted sector, or g -*twisted* V -*module*. It is an important conjecture, invariably assumed in the physics literature, that there is a

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unique simple g -twisted V -module (assuming that V is holomorphic). This is known in certain cases (eg. [D2] and [DM1]), but remains open in general. Let us assume that it holds for now, and set

$$V^* = \bigoplus_{g \in G} M(g) \quad (1.1)$$

where $M(g)$ is the postulated unique simple g -twisted V -module. In particular $M(1) = V$ where 1 is the identity of G . Our work is concerned with an analysis of the sequence

$$V^* \supseteq V \supseteq V^G. \quad (1.2)$$

It is straightforward to see that V^* is a V^G -module, so we may define the category $\mathcal{V}^*(G)$ to be the V^G -module category whose objects are V^G -submodules of direct sums of copies of V^* , and whose morphisms are V^G -homomorphisms of V^G -modules. As a special case of our results we have

Theorem 1 *Let V be a simple VOA and G a faithful, finite nilpotent group of automorphisms of V . Assume that for every $g \in G$, there is a unique simple g -twisted V -module. Then there is an equivalence of categories*

$$\phi : \mathcal{V}^*(G) \rightarrow \text{Mod-}D_\alpha(G). \quad (1.3)$$

We hasten to explain the notation. Following [DM2] and [M], if $C_G(g) = \{h \in G | gh = hg\}$ is the *centralizer* of g in G then there is a *projective representation* of $C_G(g)$ on $M(g)$ for each $g \in G$. This data may be described by a certain $\alpha \in H^3(\mathbb{Z}G)$, the group of Hochschild 3-cocycles on the integral group ring $\mathbb{Z}G$. We can *twist* the quantum double $D(G)$ by α to obtain another semi-simple algebra $D_\alpha(G)$, and Theorem 1 identifies a certain category of V^G -modules with the category $\text{Mod-}D_\alpha(G)$ of $D_\alpha(G)$ -modules. It remains to show that $\mathcal{V}^*(G)$ is the complete category of V^G -modules, and that $D_\alpha(G)$ is the same as $D^c(G)$.

Note that the origins of $D_\alpha(G)$ are quite different from those of the algebra $D^c(G)$ mentioned above. But in addition to the fact that tensor products of twisted modules are not presently understood, the algebra $D_\alpha(G)$ is a very natural object to study in

the present context. For we will show that, quite generally, the space V^* is naturally a module over $D_\alpha(G)$, and that also $D_\alpha(G)$ commutes with action of the vertex operators $Y(v, z)$ for $v \in V^G$. Thus one may loosely say that V^* is a $D_\alpha(G) \otimes V^G$ -module. On the other hand, we treat $D_\alpha(G)$ solely as an associative algebra – any quasi-quantum group structure that is available is not relevant to the present considerations.

The precise nature of the equivalence ϕ of Theorem 1 is perhaps as important as its existence. In case G is nilpotent, we will prove that V^* decomposes into a direct sum

$$V^* = \oplus_\chi M_\chi \otimes V_\chi \quad (1.4)$$

where in (1.4), χ ranges over the simple characters of $D_\alpha(G)$, M_χ is a module over $D_\alpha(G)$ which affords χ , and V_χ is a certain simple V^G -module. Then the map ϕ in (1.3) is just the extension of a *bijection*

$$\phi : M_\chi \rightarrow V_\chi \quad (1.5)$$

from simple $D_\alpha(G)$ -modules to simple V^G -modules which are contained in V^* .

From this one can see that $D_\alpha(G)$ is precisely the *grade-preserving commuting algebra* of V^G on V^* , that is $D_\alpha(G)$ is the algebra of operators on V^* which commute with the VOA V^G and which preserve the conformal grading on V^* . Thus $D_\alpha(G)$ and V^G behave in many ways like a pair of mutually commuting algebras, or a dual pair in representation theory. From this point of view, the decomposition (1.4) and bijective correspondence (1.5) take on a somewhat classical air.

We should also say that we certainly believe that these results hold for *arbitrary* finite groups G . At the moment we are unable to establish the general case, but the reader may see that at the cost of more technical detail but with no further new ideas, Theorem 1 can be established for any *solvable* group. But to keep the main ideas as clear as possible we limit our discussion to nilpotent groups, which may be considered as the first broad class of groups beyond the abelian groups.

In addition we have the following result which we certainly cannot prove as yet even for the general solvable group!

Theorem 2 *Let V be a simple VOA and let G be a faithful, finite nilpotent group of automorphisms of V . Then there is a Galois correspondence between subgroups of G and sub VOAs of V which contain V^G given by the map $H \mapsto V^H$.*

This result was established in [DM2] for G abelian (and G dihedral). We expect that some sort of duality relating the various terms of (1.2) ought to relate Theorems 1 and 2 more closely. We also pointed out in [DM2] that the theory of certain Von Neumann algebras, whose relation to VOA theory has been remarked on before (e.g., [MS]), also possesses a Galois theory (cf. [J]) and references therein). Our earlier comments on commuting algebras suggest that there may well be a close analogy between these two theories, at least with regard to the sort of questions we are studying here.

The proofs of both theorems proceed by induction on the order of G , so that one considers VOAs of the form V^K for various subgroups $K \subseteq G$. It is a result of [DM2] that if V is a simple VOA then so too is V^K , so that this assumption works well in an inductive setting. On the other hand, it is no longer appropriate to assume the unicity of simple g -twisted V -modules, and we therefore merely assume that there is a family of twisted sectors which behave in the ‘correct’ way (cf. Hypothesis 3.3). In this generality, our main result (Theorem 6.1), will also apply to rational VOAs as well as holomorphic VOAs.

The paper proceeds as follows: in Section 2, we review some basic facts from VOA theory, in particular so called *duality*, which plays an important role. Section 3 explains the role of Hochschild cohomology, following [M]. In Section 4 we describe the action of $D_\alpha(G)$ on V^* . In Section 5 we present the Zhu algebra $A(V)$ together with some variations which are adapted to orbifold theory. These ideas are developed in [DLM]. Sections 6 and 7 provide the proofs of the two theorems.

2 Vertex operator algebras and modules

In this section we recall the definitions of vertex operator algebras and modules (cf. [B], [FLM], [DM2]). We also discuss duality for twisted modules.

A *vertex operator algebra* (or VOA) (cf. [B], [FLM]) is a \mathbb{Z} -graded vector space $V = \coprod_{n \in \mathbb{Z}} V_n$ such that $\dim V_n < \infty$ and $V_n = 0$ if n is sufficiently small, equipped with a linear map

$$\begin{aligned} V &\rightarrow (\text{End } V)[[z, z^{-1}]] \\ v &\mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } V) \end{aligned} \quad (2.1)$$

and with two distinguished vectors $\mathbf{1} \in V_0$, $\omega \in V_2$ satisfying the following conditions for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large;} \quad (2.2)$$

$$Y(\mathbf{1}, z) = 1; \quad (2.3)$$

$$Y(v, z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z \rightarrow 0} Y(v, z)\mathbf{1} = v; \quad (2.4)$$

$$\begin{aligned} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1) Y(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2) Y(u, z_1) \\ = z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2) \end{aligned} \quad (2.5)$$

(Jacobi identity) where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the algebraic formulation of the δ -function at 1, and all binomial expressions are to be expanded in nonnegative integral powers of the second variable;

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\text{rank } V) \quad (2.6)$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \quad (2.7)$$

and

$$\text{rank } V \in \mathbb{Q}; \quad (2.8)$$

$$L(0)v = nv = (\text{wt } v)v \quad \text{for } v \in V_n \ (n \in \mathbb{Z}); \quad (2.9)$$

$$\frac{d}{dz} Y(v, z) = Y(L(-1)v, z). \quad (2.10)$$

This completes the definition. Note that (2.6) says that the operators $L(n)$ generate a copy of the Virasoro algebra, represented on V with central charge $\text{rank } V$. We denote the

vertex operator algebra just defined by $(V, Y, \mathbf{1}, \omega)$ (or briefly, by V). The series $Y(v, z)$ are called *vertex operators*.

Let $(V, Y, \mathbf{1}, \omega)$ are vertex operator algebra. An *automorphism* of V is a linear map $g: V \rightarrow V$ satisfying

$$gY(v, z)g^{-1} = Y(gv, z), \quad v \in V \quad (2.11)$$

$$g\mathbf{1} = \mathbf{1}, \quad g\omega = \omega. \quad (2.12)$$

Let $\text{Aut}(V)$ denote the group of all automorphisms of V .

Now each g commutes with the component operators $L(n)$ of ω , and in particular g preserves the homogeneous spaces V_n which are the eigenspaces for $L(0)$. So each V_n is a representation module for $\text{Aut}(V)$.

Let g be an automorphism of the VOA V of order N . Following [FFR] and [D2], a *weak g -twisted module* M for V is a \mathbb{C} -graded vector space $M = \coprod_{n \in \mathbb{C}} M_n$ such that for $\lambda \in \mathbb{C}$, $M_{n+\lambda} = 0$ for $n \in \frac{1}{N}\mathbb{Z}$ is sufficiently small. Moreover there is a linear map

$$\begin{aligned} V &\rightarrow (\text{End } M)[[z^{1/N}, z^{-1/N}]] \\ v &\mapsto Y_g(v, z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M) \end{aligned} \quad (2.13)$$

satisfying axioms analogous to (2.2)-(2.3) and (2.5)-(2.10). To describe these, let $\eta = e^{2\pi i/N}$ and set $V^j = \{v \in V | gv = \eta^j v\}$, $0 \leq j \leq N-1$. Thus we have a direct sum decomposition

$$V = \coprod_{j \in \mathbb{Z}/N\mathbb{Z}} V^j. \quad (2.14)$$

Then we require that for $u, v \in V$, $w \in M$,

$$Y_g(v, z) = \sum_{n \in j/N + \mathbb{Z}} v_n z^{-n-1} \quad \text{for } v \in V^j; \quad (2.15)$$

$$u_n w = 0 \quad \text{for } n \text{ sufficiently large}; \quad (2.16)$$

$$Y_g(\mathbf{1}, z) = 1; \quad (2.17)$$

$$\begin{aligned} z_0^{-1} \delta\left(\frac{z_1 - z_2}{z_0}\right) Y_g(u, z_1) Y_g(v, z_2) - z_0^{-1} \delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_g(v, z_2) Y_g(u, z_1) \\ = z_2^{-1} \left(\frac{z_1 - z_0}{z_2}\right)^{-k/N} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_g(Y(u, z_0)v, z_2) \end{aligned} \quad (2.18)$$

for $u \in V^k$. Finally, (2.6)-(2.10) go over unchanged except that in (2.9) we replace v by $w \in M$. This completes the definition. We denote this module by (M, Y_g) , or briefly by M .

Remark 2.1 A g -twisted V -module is a weak g -twisted module such that each homogeneous subspace M_n is finite-dimensional. A g -twisted V -module is a V -module if $g = 1$. Moreover, a g -twisted V -module restricts to an ordinary V^0 -module.

Remark 2.2 For a weak g -twisted V -module M and $\lambda \in C$, $M(\lambda) = \sum_{n \in \frac{1}{N}\mathbb{Z}} M_{\lambda+n}$ is a weak g -twisted submodule of V by (2.15). It is clear that $M(\lambda) = M(\mu)$ if and only if $\lambda - \mu \in \frac{1}{N}\mathbb{Z}$. Moreover, $M = \coprod_{\lambda \in C/\frac{1}{N}\mathbb{Z}} M(\lambda)$. Thus it is enough to study weak g -twisted module of type

$$M = \coprod_{n \in \frac{1}{N}\mathbb{Z}, n \geq 0} M_{c+n}$$

where $c \in \mathbb{C}$ is a fixed and $M_c \neq 0$. We call M_c the *top level* of M . Note that M has this restricted gradation if M is irreducible. The same comments also apply to g -twisted modules and ordinary modules.

Next we shall present the duality properties for twisted modules. Let V be a VOA, $g \in \text{Aut}(V)$ of order N and $W = \coprod_{n \geq n_0} W_n$ a g -twisted V -module. Let W_n^* be the dual space of W_n and $W' = \coprod_{m \geq m_0} W_m^*$ the restricted dual of W . We denote by $\langle \cdot, \cdot \rangle : W' \times W \rightarrow \mathbb{C}$ the restricted paring such that $\langle W_n^*, W_m \rangle = 0$ unless $m = n$.

Set

$$\mathbb{C}[(az_1 + bz_2)^{1/N}, (az_1 + bz_2)^{-1/N} | a, b \in \mathbb{C}, ab \neq 0].$$

For each of the two orderings (i_1, i_2) of the set $\{1, 2\}$ there is an injective ring map

$$\iota_{i_1 i_2} : R \rightarrow \mathbb{C}[[z_1^{1/N}, z_1^{-1/N}, z_2^{1/N}, z_2^{-1/N}]]$$

by which an element $(az_1 + bz_2)^n \in R$ for $n \in \frac{1}{N}\mathbb{Z}$ is expanded in nonnegative integral powers of z_{i_2} . Using the proof of the “duality” for a generalized vertex algebra given in Chapter 9 (Propositions 9.12 and 9.13) of [DLe] we have

Proposition 2.3 (i) **Rationality:** Let $u \in V^r, v \in V^s, w \in W, w' \in W'$ with $0 \leq r, s < N$ and $r, s \in \mathbb{Z}$. Then there is $f \in R$ of the form $f(z_1, z_2) = h(z_1, z_2)/z_1^{r/N+m} z_2^{s/N+n} (z_1 - z_2)^t$ with $h(z_1, z_2) \in \mathbb{C}[z_1, z_2]$ and $m, n, t \in \mathbb{Z}$ nonnegative such that t only depends on u and v and that

$$\langle w', Y(u, z_1)Y(v, z_2)w \rangle = \iota_{12}f(z_1, z_2). \quad (2.19)$$

(ii) **Commutativity:** If u, v, w, w', f are as in (i) then we also have

$$\langle w', Y(v, z_2)Y(u, z_1)w \rangle = \iota_{21}f(z_1, z_2). \quad (2.20)$$

(iii) **Associativity:** In the same notation,

$$\langle w', Y(Y(u, z_0)v, z_2)w \rangle = \iota_{20}f(z_0 + z_2, z_2). \quad (2.21)$$

Using rationality (i) and associativity (iii) and following the proof of Proposition 4.1 of [DM2] we have (see also Lemma 5.5 of [DLi]):

Proposition 2.4 Let V be a vertex operator algebra and g an automorphism of order N . Assume that M is a weak g -twisted module generated by $S \subset M$ in the sense that M is the linear span of

$$v_{n_1}^1 \cdots v_{n_k}^k s$$

for $v^i \in V, n_i \in \frac{1}{N}\mathbb{Z}, s \in S$ and $k \geq 0$. Then M is the linear span of the subset $\{u_n s | u \in V, n \in \frac{1}{N}\mathbb{Z}, s \in S\}$. In particular, if M is simple then for each $0 \neq w \in M$, M is spanned as weak g -twisted V -module by $u_n v$ for $u \in V, n \in \frac{1}{N}\mathbb{Z}$. \square

3 Group cohomology and VOAs

Let $\text{Irr}(V)$ be the set of (isomorphism classes of) inequivalent simple V -modules. Following Section 2 of [DM1], there is a *right* action of G on $\text{Irr}(V)$ (we are taking G to be a subgroup of $\text{Aut}(V)$) given as follows: If $M = (M, Y_M) \in \text{Irr}(V)$ and $g \in G$ then

$$(M, Y_M) \circ g = (M \circ g, Y_M \circ g) \quad (3.1)$$

where by definition $M \circ g = M$ and

$$Y_M \circ g(v, z) = Y_M(gv, z). \quad (3.2)$$

More generally, let $\text{Irr}_g(V)$ be the set of inequivalent simple g -twisted V -modules, $g \in G$. Then (3.1) and (3.2) define maps, for $h \in G$,

$$\text{Irr}_g(V) \times \{h\} \rightarrow \text{Irr}_{h^{-1}gh}(V).$$

Thus if \mathcal{M} is the union of all $\text{Irr}_g(V)$, $g \in G$, we have a right action $\mathcal{M} \times G \rightarrow \mathcal{M}$ of G on \mathcal{M} . Of course these definitions hold true for *any* g -twisted V -module: there is a right G -action which preserves the set of simple objects.

Next let V be a VOA with $g \in \text{Aut}(V)$, and let $M = (M, Y_M)$ be a g -twisted V -module. An *automorphism* of M consists of a pair $(x, \alpha(x))$ satisfying the following: $x : M \rightarrow M$ and $\alpha(x) : V \rightarrow V$ are linear isomorphisms such that

$$\begin{aligned} xY_M(v, z)x^{-1} &= Y_M(\alpha(x)v, z) \\ \alpha(x)g &= g\alpha(x), \alpha(x)\mathbf{1} = \mathbf{1}, \alpha(x)\omega = \omega \end{aligned} \quad (3.3)$$

for $v \in V$. If V and M are both simple it is easy to see from the axioms that the following hold: $x \rightarrow \alpha(x)$ is a group homomorphism α with kernel consisting of all scalar operators on M . Moreover the image of α is a group of automorphisms of V that commutes with g . Thus x is then just an isomorphism from (M, Y_M) to $(M, Y_M) \circ \alpha(x)$.

This approach is basically the opposite of that in [DM1] for constructing projective representations on twisted sectors. We quickly recall the details.

Let $(M, Y_M) \in \text{Irr}_g(V)$ and let H be the subgroup of $\text{Aut}(V)$ which commutes with g and which satisfies $(M, Y_M) \circ h \cong (M, Y_M)$, all $h \in H$. This means that there is a linear isomorphism $\phi(h) : M \rightarrow M$ satisfying

$$\phi(h)Y_M(v, z)\phi(h)^{-1} = Y_M \circ h(v, z) = Y_M(hv, z) \quad (3.4)$$

for $v \in V$. Thus $(\phi(h), h)$ is an automorphism of M , and as explained in Section 2 of [DM1], the simplicity of M together with Schur's lemma shows that $h \mapsto \phi(h)$ is a projective representation of H . In effect, ϕ is a section of the group homomorphism α .

In general, given H and ϕ above, we denote by \hat{H} the central extension of H obtained from ϕ . When necessary we let α_g (not to be confused with α !) be the corresponding 2-cocycle in $C^2(H, \mathbb{C}^\times)$.

We next consider the action of $g \in \text{Aut}(V)$ of finite order on $\text{Irr}_g(V)$. We will show that it is trivial.

Lemma 3.1 *Let $(M, Y_M) \in \text{Irr}_g(V)$. Then $(M, Y_M) \circ g \cong (M, Y_M)$.*

Proof: Let M_c be the top level of M . Thus $M_c \neq 0$ while $M_{c+n} = 0$ for $n < 0$. Then from Remark 2.2 we have

$$M = \prod_{n=0}^{\infty} M_{c+\frac{n}{N}}$$

where N is the order of g . Define $\phi(g) : M \rightarrow M$ as follows:

$$\phi(g)|_{M_{c+\frac{n}{N}}} = e^{-2\pi i n/N}. \quad (3.5)$$

From (3.4) we see that $(\phi(g), g)$ is an automorphism of (M, Y_M) . The lemma follows. \square

We continue with a simple VOA V , $g \in \text{Aut}(V)$ of finite order N , and $(M, Y_M) \in \text{Irr}_g(V)$. Let H be as above. Thus H is a group which contains g . Let $h \rightarrow \phi(h)$ be the projective representation of H given by (3.4) and let \hat{H} be the central extension of H (by \mathbb{C}^\times) which acts on M .

We let $\phi(g)$ be the map (3.5), regarded as an element of \hat{H} . So $(\phi(g), g)$ is an automorphism of M .

Lemma 3.2 *$\phi(g)$ lies in the center of \hat{H} .*

Proof: Let $\mathbb{C}^\times \cong X \leq Z(\hat{H})$ be such that \hat{H} is the central extension $1 \rightarrow X \rightarrow \hat{H} \rightarrow H \rightarrow 1$. Since $g \in Z(H)$ then clearly we have $X\langle\phi(g)\rangle \leq \hat{H}$ and $[\phi(g), \hat{H}] \leq X$.

On the other hand on the top level M_c of M , $\langle\phi(g)\rangle$ is the subgroup of $X\langle\phi(g)\rangle$ acting trivially. Then $\langle\phi(g)\rangle \leq \hat{H}$, so that $[\phi(g), \hat{H}] \leq X \cap \langle\phi(g)\rangle = 1$. The lemma follows. \square

The following axiomatizes one of the basic situation with which we are concerned.

Hypothesis 3.3 *V is a simple VOA and $G \leq \text{Aut}(V)$ a finite group of automorphisms. For each $g \in G$ there is $(M(g), Y_g) \in \text{Irr}_g(V)$ such that, for all $h \in G$,*

$$(M(g), Y_g) \circ h \cong (M(h^{-1}gh), Y_{h^{-1}gh}). \quad (3.6)$$

Remark 3.4 *It is a well-known conjecture that (3.6) always holds if V is holomorphic, $M(g)$ being the unique element of $\text{Irr}_g(V)$ in that case.*

For the rest of the paper we will concentrate on the element α of

$$HH^3(\mathbb{Z}G) \simeq \bigoplus_* H^2(C_G(g), \mathbb{C}^\times) \quad (3.7)$$

(where HH stands for *Hochschild* cohomology and the sum in (3.7) runs over one g in each conjugacy class of G) determined by Hypothesis 3.3. Specifically, for $g \in G$, let $\alpha_g \in C^2(C_G(g), \mathbb{C}^\times)$ be the 2-cocycle corresponding to the projective representation $\phi = \phi_g$ (3.4).

More precisely, using (3.6), we have maps $\phi_g(h): M(g) \rightarrow M(hgh^{-1})$ for $g, h \in G$ and

$$\phi_g(h)Y_g(v, z)\phi_g(h)^{-1} = Y_{hgh^{-1}}(hv, z). \quad (3.8)$$

Compatibility yields

$$\phi_g(hk) = \alpha_g(h, k)^{-1} \phi_{kgk^{-1}}(h) \phi_g(h) \quad (3.9)$$

for some $\alpha_g(h, k) \in \mathbb{C}^\times$ (even S^1) satisfying

$$\alpha_g(hk, l) \alpha_{lgl^{-1}}(h, k) = \alpha_g(h, kl) \alpha_g(k, l). \quad (3.10)$$

If $h, k, l \in C_G(g)$ then (3.10) reduces to the assertion that $\alpha_g \in C^2(C_G(g), \mathbb{C}^\times)$; it is the 2-cocycle associated with ϕ_g .

4 The twisted quantum double

We continue to assume Hypothesis 3.3. Associated to this situation is the *algebraic* space

$$V^* = \bigoplus_{g \in G} M(g). \quad (4.1)$$

We will define a certain associative algebra $D_\alpha(G)$, the *twisted quantum double*, where α is the element in $HH^3(\mathbb{Z}G)$ discussed in Section 3. We will then prove

Theorem 4.1 *The following hold:*

- (i) V^* is a module over $D_\alpha(G)$.
- (ii) Every simple $D_\alpha(G)$ -module occurs as a submodule of V^* .
- (iii) If V^G is the sub VOA of G -invariants of V then V^* is a V^G -module and the actions of $D_\alpha(G)$ and V^G on V^* commute.

Introduce the complex group algebra $\mathbb{C}[G]$ and its dual $\mathbb{C}[G]^*$ with basis $e(g)$, $g \in G$ satisfying $e(g)e(h) = e(g)\delta_{g,h}$. The group of units of $\mathbb{C}[G]^*$ is the multiplicative group

$$U = \left\{ \sum_{g \in G} a_g e(g) \mid a_g \in \mathbb{C}^\times \right\}. \quad (4.2)$$

The group G acts on $\mathbb{C}[G]^*$ on the right via $e(g) \cdot h = e(h^{-1}gh)$. This action preserves U , which thereby becomes a multiplicative right $\mathbb{Z}G$ -module.

Define $\alpha : G \times G \rightarrow U$ via

$$\alpha(h, k) = \sum_{g \in G} \alpha_g(h, k) e(g). \quad (4.3)$$

Then (3.10) is equivalent to

$$\alpha(hk, l) \alpha(h, k)^l = \alpha(h, kl) \alpha(k, l) \quad (4.4)$$

which says that $\alpha \in C^2(\mathbb{Z}G, U)$, the group of 2-cocycles on G with values in U . We leave it to the reader for sort-out the relation between $H^2(G, U)$ and $HH^3(\mathbb{Z}G)$.

Following [M], the *twisted quantum double* $D_\alpha(G)$ has underlying space $\mathbb{C}[G] \otimes \mathbb{C}[G]^*$ with multiplication

$$a \otimes e(x) \cdot b \otimes e(y) = \alpha_y(a, b) ab \otimes e(b^{-1}xb)e(y). \quad (4.5)$$

If $\alpha = 1$ this reduces to the usual quantum double [Dr]. $D_\alpha(G)$ is a semi-simple associative algebra.

If $m \in M(g)$, $g \in G$, we define

$$a \otimes e(x) \cdot m = \delta_{x,g} \phi_x(a)m \quad (4.6)$$

where $\phi_g(a) : M(g) \rightarrow M(aga^{-1})$ is as before. Using (3.9) and (4.5) shows that (4.6) defines a left action of $D_\alpha(G)$ on V^* . This proves (i).

Part (ii) is easy. We have already pointed out in Section 2 that g -twisted modules are ordinary V^G -modules. So certainly V^* is a V^G -module.

Using (3.8) and (4.6) we find that for $v \in V$ we have

$$a \otimes e(x) \bar{Y}(v, z) = \bar{Y}(av, z) a \otimes e(x) \quad (4.7)$$

as operators on V^* , where we have used $\bar{Y}(v, z)$ to denote the operator on V^* which acts on $M(g)$ as $Y_g(v, z)$. In particular, it is clear that if $v \in V^G$ then $a \otimes e(x)$ commutes with $\bar{Y}(v, z)$. So (iii) of the theorem holds.

We turn our attention to (ii). Fix $g \in G$ and set $C = C_G(g)$. If $a, b \in C$ then from (4.5) we see that $D_\alpha(G)$ contains the α_g -twisted group algebra $\mathbb{C}_{\alpha_g}[G]$ which has basis indexed by $a \in C$ and multiplication $a \cdot b = \alpha_g(a, b)ab$. More precisely, if $S(g)$ is the subspace of $D_\alpha(G)$ with basis $a \otimes e(g)$ for $a \in C$, then $S(g)$ is isomorphic to $\mathbb{C}_{\alpha_g}[C]$ via $a \otimes e(y) \mapsto a$.

Now (4.6) defines a left action of $S(g)$ on $M(g)$. If we also let $D(g)$ be the subspace of $D_\alpha(G)$ spanned by $a \otimes e(g)$ for all $a \in G$ then $D(g)$ is an $D_\alpha(G) - S(g)$ -bimodule.

Lemma 4.2 *Let K be the conjugacy class of G that contains g . Then there is an isomorphism of $D_\alpha[G]$ -modules:*

$$D(g) \otimes_{S(g)} M(g) \simeq \oplus_{h \in K} M(h)$$

given by $a \otimes e(g) \otimes m \rightarrow \phi_g(a)m$ for $a \in G$. \square

This is an easy calculation. The point is that it is shown in [M] (see also [DPR]) that the simple $D_\alpha(G)$ -modules are precisely the modules $D(g) \otimes_{S(g)} X$ where X ranges over the simple $S(g)$ -modules and g ranges over one element in each conjugacy class of G . Thus after Lemma 4.2, Theorem 4.1 (ii) is a consequence of

Lemma 4.3 *Every simple $S(g)$ -module is a submodule of $M(g)$.*

We will need

Lemma 4.4 *For $m \geq 1$ let u_1, \dots, u_m and w_1, \dots, w_m be non-zero vectors in V and linearly independent vectors in $M(g)$ respectively. Then*

$$\sum_{i=1}^m Y_g(u_i, z)v_i \neq 0.$$

Proof: This is the same as the proof of Lemma 3.1 of [DM2]. We just have to replace the duality arguments for simple V -modules used in [DM2] by the corresponding duality statements for twisted modules as stated in Section 2. \square

Proof of Lemma 4.3: We use Theorem 2 of [DM2] which tells us that every simple $\mathbb{C}[C]$ -module is a submodule of V . Let $S = S(g)$.

Let A be a simple S -module which is a submodule of $M(g)$ and let B be any simple S -module. We can find a simple $\mathbb{C}[C]$ -module D such that $\text{Hom}_S(A^* \otimes B, D) \neq 0$. Thus B is a submodule of the S -module $D \otimes A$.

We may take $D \subset V$, so that expressions of the form $\sum_{i=1}^m Y_g(v_i, z)w_i$ with $v_i \in D$ and $w_i \in A$ can be used. We can thus complete the proof using Lemma 4.4 as in the proof of Theorem 2 of [DM2]. \square

5 Zhu algebras

In [Z], Zhu introduced an associative algebra $A(V)$ associated to a VOA V which is extremely useful in studying the representation theory of V . There are analogues for g -twisted modules which are developed in [DLM], which we review here.

Fix a VOA V and an automorphism g of V (of finite order). For $u, v \in V$ with u homogeneous, define a product $u * v$ as follows:

$$u * v = \text{Res}_z Y(u, z) \frac{(z+1)^{\text{wt } u}}{z} v = \sum_{i=0}^{\infty} \binom{\text{wt } u}{i} u_{i-1} v. \quad (5.1)$$

Then extends (5.1) to a linear product $*$ on V .

For $0 < r \leq 1$ let V^r the eigenspace of g with eigenvalue $e^{2\pi ir}$, that is,

$$V^r = \{v \in V | gv = e^{2\pi ir} v\}. \quad (5.2)$$

Define a subspace $O_g(V)$ of V to be the linear span of all elements $u \circ_g v$ of the following type if u is homogeneous, $u \in V^r$ and $v \in V$:

$$u \circ_g v = \begin{cases} \text{Res}_z Y(u, z) \frac{(z+1)^{\text{wt } u}}{z^2} v, & \text{if } r = 1 \\ \text{Res}_z Y(u, z) \frac{(z+1)^{\text{wt } u+r-1}}{z} v, & \text{if } r < 1. \end{cases} \quad (5.3)$$

Note that if $r < 1$ then $u \circ_g \mathbf{1} = u$. Thus we have

Lemma 5.1 *If $V^{\langle g \rangle}$ is the sub VOA of g -invariants of V then $V = V^{\langle g \rangle} + O_g(V)$. \square*

Let (M, Y_g) be a weak g -twisted V -module. For homogeneous $u \in V$, the component operator $u_{\text{wt } u-1}$ preserves each homogeneous subspace of M and in particular acts on the top level M_c of M (cf. Section 2). Let $o_g(u)$ be the restriction of $u_{\text{wt } u-1}$ to M_c , so that we have a linear map

$$\begin{aligned} V &\rightarrow \text{End}(M_c) \\ u &\mapsto o_g(u). \end{aligned} \quad (5.4)$$

Note that if $u \in V^r$ with $r < 1$ then $o_g(u) = 0$ from (2.15).

Set $A_g(V) = V/O_g(V)$. Then we have [DLM]:

Theorem 5.2 *(i) $A_g(V)$ is an associative algebra with multiplication induced from $*$ and the centralizer $C(g)$ of g in $\text{Aut}(V)$ induces a group of algebra automorphisms of $A_g(V)$.*

(ii) $u \mapsto o_g(u)$ gives a representation of $A_g(V)$ on M_c . Moreover, if every weak g -twisted module is completely reducible, $A_g(V)$ is semisimple.

(iii) $M \mapsto M_c$ gives a bijection between the set of equivalence classes of simple weak g -twisted V -modules and the set of equivalence classes of simple $A_g(V)$ -modules. \square

Next, with V as before and with G an automorphism group of V , we define

$$O_G(V) = \bigcap_{g \in G} O_g(V), \quad A_G(V) = V/O_G(V). \quad (5.5)$$

Lemma 5.3 $A_G(V)$ is an associative algebra with respect to product $*$.

Proof: From Theorem 5.2 (i), $O_g(V)$ is a 2-sided ideal of V with respect to $*$. Moreover if $u, v, w \in V$ then $(u * v) * w - u * (v * w) \in O_g(V)$ for all $g \in G$. The lemma follows. \square

From Theorem 5.2 we see that the simple modules for $A_G(V)$ correspond to the top levels of weak simple g -twisted V -modules for all $g \in G$.

We will need to consider the algebra $A_G(V)$, in a special case, in the course of the proof of Theorem 6.1.

Theorem 5.4 Let V be a VOA and G a finite group of automorphisms of V with $g \in G$ in the center of G . Let M, N be two simple g -twisted V -modules with X, Y the top levels of M and N respectively. Assume that the weight of Y is less than or equal to the weight of X . Let N' be the V^G -submodule of N generated by Y . Exactly one of the following holds:

- (i) $M \circ h \simeq N$ (isomorphism of g -twisted V -modules) for some $h \in G$.
- (ii) $\text{Hom}_{V^G}(N', M) = 0$.

Proof: As $g \in Z(G)$ then G induces algebra automorphisms of $A_g(V)$. Assume that (i) fails. So N is not isomorphic to $M \circ h$ for any $h \in G$.

Let $M = M_1, \dots, M_r$, $N = N_1, \dots, N_s$ be the distinct conjugates of M and N under the action of G , and let $X = X_1, \dots, X_r$, $Y = Y_1, \dots, Y_s$ be the corresponding top levels. So these $r + s$ spaces afford inequivalent simple $A_g(V)$ -modules (Theorem 5.2), and on their direct sum $A_g(V)$ realizes the algebra $\oplus_{i=1}^r \text{End}(X_i) \oplus \oplus_{j=1}^s \text{End}(Y_j)$.

One computes that the algebra C of G -invariants on $\oplus_{i=1}^r \text{End}(X_i)$ is non-zero and satisfies $\text{Ann}_C(Y) = C$, $\text{Ann}_X(C) = 0$. Note that the image of the natural map $A(V^G) \rightarrow A_g(V)$ contains C . We conclude that $\text{Hom}_{A(V^G)}(Y, X) = 0$.

But if (ii) of the Theorem fails then there is $f \in \text{Hom}_{V^G}(N', M)$ satisfying $0 \neq f(N') \subset M$. As Y generates N' then $f(Y) \neq 0$, and by the assumption that the weight of Y is less than or equal to weight of X we conclude that $0 \neq f(Y) \subset X$. Thus f induces a nonzero element of $\text{Hom}_{A(V^G)}(Y, X)$, which is the desired contradiction. \square

6 Proof of Theorem 1

We continue with the assumptions and notation of Hypothesis 3.3. As described in Section 3, there is a projective representation of $C_G(g)$ on $M(g)$ for $g \in G$, corresponding to a 2-cocycle $\alpha_g \in C^2(C_G(g), S^1)$. In this way $M(g)$ becomes a module over the twisted group algebra $\mathbb{C}_{\alpha_g}[C_G(g)]$. Moreover this algebra commutes with the vertex operators of V^G .

Theorem 6.1 *Assume that G is nilpotent. Then the following hold:*

(A) *There is a decomposition of $M(g)$ into simple modules over $\mathbb{C}_{\alpha_g}[C_G(g)] \otimes V^G$ of the form*

$$M(g) \simeq \oplus_{\chi_g} M_{\chi_g} \otimes V_{\chi_g} \quad (6.1)$$

where χ_g ranges over the simple characters of $\mathbb{C}_{\alpha_g}[C_G(g)]$, M_{χ_g} is a module over $\mathbb{C}_{\alpha_g}[C_G(g)]$ which affords χ_g , and V_{χ_g} is a simple V^G -module.

(B) *Let $g, h \in G$. Then there is an isomorphism of V^G -modules $V_{\chi_g} \simeq V_{\psi_h}$ if and only if there is $k \in G$ such that $k(\chi_g) = \psi_h$.*

Remark 6.2 *The notation $k(\chi_g) = \psi_h$ means that $h = kgk^{-1}$ and also if $z \in C_G(g)$ then $\chi_g(z) = \psi_h(kzk^{-1})$.*

Let us explain how Theorem 6.1 implies Theorem 1. As explained in Section 4, if we fix a choice of g in each conjugacy class of G then we have

$$V^* = \oplus_g \text{Ind}_{S(g)}^{D(g)} M(g). \quad (6.2)$$

Here, we are using the notation of Section 4 (cf. Lemma 4.2) and (6.2) is an isomorphism of $D_\alpha(G)$ -modules. Since $D_\alpha(G)$ commutes with V^G on V^* , it follows from (6.1) and results from Section 4 that V^* is a $D_\alpha(G) \otimes V^G$ -module with decomposition into simple modules

$$V^* = \oplus_g (\text{Ind}_{S(g)}^{D(g)} M(g)) \otimes V_{\chi_g}. \quad (6.3)$$

Moreover *every* simple $D_\alpha(G)$ -module occurs in (6.3) with non-zero multiplicity. Thus the map

$$\text{Ind}_{S(g)}^{D(g)} M(g) \xrightarrow{\phi} V_{\chi_g}$$

induces a map

$$\{\text{simple } D_\alpha(G)\text{-modules}\} \xrightarrow{\phi} \{\text{simple } V^G\text{-modules}\}.$$

Theorem 6.1 (B) tells us that ϕ is an injection, so that ϕ is a bijection from the set of simple $D_\alpha(G)$ -modules to $\text{Im}\phi$, which is just the set of simple V^G -modules contained in V^* (for by Theorem 6.1 (A), V^* is completely reducible as V^G -module). Thus we have constructed the bijection ϕ of (1.5). And again since the categories $\mathcal{V}^*(G)$ and $\text{Mod-}D_\alpha(G)$ are semi-simple then ϕ extends to the equivalence of categories (1.3). So then Theorem 1 is proved.

Note, in fact, that we have proved Theorem 1 for any simple VOA under the assumptions of Hypothesis 3.3. We thus expect (cf. Section 2 of [DM1]) that our results will apply to any simple rational VOA V for which G is a group of *inner* automorphism of V (cf. [DM1] for the definition of inner automorphism).

We record some standard facts about finite nilpotent group (see [G], for example) which we often use without comment.

Lemma 6.3 *Let G be a finite nilpotent group. Then*

- (i) *Every maximal subgroup of G is normal and of prime index in G .*
- (ii) *Subgroups and quotient groups of G are nilpotent.*
- (iii) *Central extensions of G are nilpotent.*
- (iv) *If χ is a simple character of G of degree greater than one, then there is a maximal subgroup H of G and a simple character ψ on H such that $\chi = \text{Ind}_H^G(\psi)$. \square*

We now begin the proof of Theorem 6.1, using induction on $|G|$. If $|G| = 1$ there is nothing to prove. We frequently use the following facts: if $H \leq G$ then the invariant sub VOA V^H is itself simple (Theorem 4.4 of [DM2]); moreover if $H \triangleleft G$ then G/H is a *faithful* group of automorphisms of V^H (Proposition 3.3 of [DM2]). Thus the assumptions about the pair (V, G) underlying Theorem 6.1 also apply to the pair $(V^H, G/H)$ if $H \triangleleft G$.

We start with the proof of (A). So fix $g \in G$ and let $C = C_G(g)$, $M = M(g)$. We must establish the decomposition (6.1) together with the simplicity of the V^G -module V_{χ_g} for each χ_g .

Case A1: C is a proper subgroup of G .

In this case let $C < H < G$ with H a maximal subgroup of G . By Lemma 6.3 (i) we have $H \triangleleft G$ and $[G : H]$ is a prime p . Since $C = C_H(g)$, induction tells us that there is a decomposition

$$M = \oplus_{\chi_g} M_{\chi_g} \otimes V'_{\chi_g} \quad (6.4)$$

where V'_{χ_g} is a simple V^H -module.

Choose $x \in G \setminus H$. A simple argument using the containments $C < H \triangleleft G$ shows that the element $x^{-1}gx$ is not conjugate to g in G . Since $M \circ x = M(x^{-1}gx)$, it follows from our induction assumption of (B) that x induces an automorphism of V^H of prime order p such that V_{χ_g} and $x(V_{\chi_g})$ are inequivalent simple V^H -modules.

Now apply Theorem 6.1 of [DM2] to conclude that the restriction of V_{χ_g} from V^H to V^G remains irreducible. Then (6.4) is the desired decomposition of M .

So we may now assume that $C = G$, that is $g \in Z(G)$. For each simple character χ of $\mathbb{C}_{\alpha_g}[G]$ let M^χ denote the χ -homogeneous component of M . We need to show, then, that

$$M^\chi = M_\chi \otimes V_\chi \quad (6.5)$$

for some simple V^G -module V_χ .

Case A2: χ has degree greater than 1.

In this case there is $H \triangleleft G$ with $[G : H] = p$, a prime, together with a simple character ψ of $\mathbb{C}_{\alpha_g}[H]$ such that χ is induced from ψ . Note that we identify α_g with its restriction to an element of $C^2(H, S^1)$. Note also that we necessarily have $g \in H$ in this situation, as g acts as a non-zero scalar on any simple $\mathbb{C}_{\alpha_g}[G]$ -module (Lemma 3.2).

By induction there is a decomposition of M as V^H -module such that, with earlier notation,

$$M^\psi = M_\psi \otimes V_\psi$$

where naturally M_ψ is the appropriate simple $\mathbb{C}_{\alpha_g}[H]$ -module, and V_ψ is a simple V^H -module.

If $x \in G \setminus H$ then we necessarily have that the characters $x^i(\psi)$ for $i = 0, \dots, p-1$ are pairwise inequivalent simple characters of $\mathbb{C}_{\alpha_g}[H]$ (cf. Lemma 6.2 of [DM2]). So again

Theorem 6.1 of [DM2] tells us, since $x(V_\psi) \simeq V_{x(\psi)}$ as V^H -modules, that the restriction of V_ψ to V^G is irreducible. So there are the following isomorphisms of V^G -modules:

$$\begin{aligned} M^\chi &= \oplus_{i=0}^{p-1} M^{x^i(\psi)} \simeq \oplus_{i=0}^{p-1} M_{x^i(\psi)} \otimes V_{x^i(\psi)} \\ &\simeq \oplus_{i=0}^{p-1} M_{x^i(\psi)} \otimes V_\psi \\ &\simeq M_\chi \otimes V_\chi. \end{aligned}$$

(in the last line we used $\chi = \text{Ind}_{\mathbb{C}_{\alpha_g}[H]}^{\mathbb{C}_{\alpha_g}[G]}(\psi) = \oplus_{i=0}^{p-1} x^i(\psi)$, and we identified V_χ with the restriction of V_ψ to V^G .) So (6.5) holds in this case.

Case A3: χ has degree 1.

In this case we need to show that M^χ itself is a simple V^G -module. This follows from Proposition 2.4 and the proof of Theorem 4.4 of [DM2].

This completes the inductive proof of (A), and it remains to prove (B). So let $g, h \in G$ be such that the two V^G -modules V_{χ_g} and V_{ψ_h} are isomorphic. We must show that $k(\chi_g) = \psi_h$ for some $k \in G$. Let $C = C_G(g)$, $D = C_G(h)$, $M = M(g)$ and $N = N(h)$. Note that V_{χ_g} and V_{ψ_h} have top levels of the same weight.

Case B1: $\langle C, D \rangle \neq G$.

As before there is $H \triangleleft G$ with $[G : H]$ a prime p and $\langle C, D \rangle < H$. Let $x \in G \setminus H$.

If V_{χ_g} is V^H -isomorphic to $x^i(V_{\psi_h})$ for some i , then by induction $k(\chi_g) = x^i(\psi_h)$ for some $k \in H$ and we are done. So we may assume that V_{χ_g} is *not* V^H -isomorphic to $x^i(V_{\psi_h})$ for any i .

Now apply Theorem 5.4 to the action of G/H on the VOA V^H to conclude that $\text{Hom}_{V^G}(V'_{\chi_g}, V_{\psi_h}) = 0$ where V'_{χ_g} is the V^G -submodule of V_{χ_g} spanned by the top level. As $V_{\chi_g} \simeq V_{\psi_h}$ as V^G -modules this is the desired contradiction.

Case B2: $\langle h, C \rangle \neq G$.

In this case g and h cannot be conjugate in G , so certainly there is no $k \in G$ with $k(\chi_g) = \psi_h$. Now proceed as in Case B1.

Case B3: $C \neq G$.

After case B2 we may assume that $C < H \triangleleft G$ with $[G : H] = p$, a prime, and $h \notin H$. In this case we get the usual decomposition (6.1) of M into V^H -modules for simple V^H -

modules V_{χ_h} . But the analogous decomposition does *not* hold for N , since $h \notin H$. There is a decomposition of N into V^G -modules, by part (A), but as a V^H -module the summands of N become \bar{h} -twisted V^H -modules, where \bar{h} is the automorphism of V^H (of order p) induced by h .

If $D_0 = D \cap H$ then $D = D_0 \langle h \rangle$, and since h lies in the center of D then the restriction of every irreducible character of D to D_0 remains irreducible. So for an irreducible character α of D_0 , $\text{Ind}_{D_0}^D(\alpha) = \sum_{i=0}^{p-1} \chi \lambda^i$ where χ is some irreducible character of D contained in $\text{Ind}_{D_0}^D(\alpha)$, and λ generates the group of characters of D/D_0 ($\simeq \mathbb{Z}_p$). From this we can see that N decompose into \bar{h} -twisted V^H -modules as follows:

$$N = \oplus_{\psi_h} (\oplus_{i=0}^{p-1} M_{\psi_h} \otimes V_{\psi_h \lambda^i}). \quad (6.6)$$

Here, ψ_h ranges over certain simple characters of $\mathbb{C}_{\alpha_h}[D]$, M_{ψ_h} is a $\mathbb{C}_{\alpha_h}[D]$ -module affording ψ_h , and λ is as before. (6.6) is supposed to mean that $\oplus_{i=0}^{p-1} M_{\psi_h} \otimes V_{\psi_h \lambda^i}$ is an \bar{h} -twisted V^H -module; the individual summands are not.

Let T be the top level of the V^H -module V_{χ_g} (6.1) and U the top level of the simple \bar{h} -twisted V^H -module generated by V_{ψ_h} . So T and U afford simple modules for the algebra $A_{G/H}(V^H)$ (cf. (5.5)).

Now h induces an automorphism of $A_{G/H}(V^H)$, and as in Case A1, since $h(V_{\chi_g}) \not\subseteq V_{\chi_g}$ as V^H -modules then T and $h(T)$ afford inequivalent $A_{G/H}(V^H)$ -modules. On the other hand $h(U) \simeq U$ since h leaves invariant each of the summands in (6.6). So T and U necessarily afford inequivalent $A_{G/H}(V^H)$ -modules. Now the argument of Theorem 6.1 of [DM2] together with Lemma 5.1 yields that T and U yield inequivalent $A(V^G)$ -modules and we are done as before.

After Case B3 we may assume that both g and h lie in $Z(G)$, i.e., $C = D = G$.

Case B4: χ_g has degree greater than 1.

Let (H, α) be such that $H \triangleleft G$, $[G : H] = p$, a prime and α is a (simple) character of $\mathbb{C}_{\alpha_g}[H]$ satisfying $\text{Ind}(\alpha) = \chi_g$. Note that g lies in the center of $\mathbb{C}_{\alpha_g}[G]$ (see Lemma 3.2) hence lies in H .

If there is $k \in H$ such that $\text{Ind}(k(\alpha)) = \psi_h$ then we are done. So we may assume that this is not the case. Now as V^H -module we have (in the notation of (6.5))

$$M^\alpha = M_\alpha \otimes V_\alpha$$

for a simple V^H -module V_α . Since $g(\alpha) \neq \alpha$ for $g \in G \setminus H$ then restriction of V_α to V^G is simple by Theorem 6.1 of [DM2] once more. Now if $h \in H$ then we are done by another application of Theorem 5.4.

So we may take $h \in G \setminus H$. In this case the argument follows the same lines, but we have to use the variation employed in case B3. We omit the straightforward details.

Case B5: $\langle g, h \rangle \neq G$.

After Case B4 we may assume that both χ_g and ψ_h have degree 1. We can complete the proof in this case using arguments of [DM2]. Namely, choose a subgroup H with $\langle g, h \rangle < H$ and H of prime index p , so that $H \triangleleft G$, and let σ, τ be the restriction of χ_g, ψ_h to $\mathbb{C}_{\alpha_g}[H]$ and $\mathbb{C}_{\alpha_h}[H]$ respectively. If we let λ generate the character group of G/H then we have $\text{Ind}(\sigma) = \bigoplus_{i=0}^{p-1} \lambda^i \chi_g$ and $\text{Ind}(\tau) = \bigoplus_{i=0}^{p-1} \lambda^i \psi_h$ (cf. Lemma 6.2 of [DM2]).

Let $V_{\lambda^i \chi_g}$ and $V_{\lambda^i \psi_h}$ be the appropriate V^G -modules in M and N respectively and let V_σ and V_τ be the corresponding V^H -modules. So we have

$$V_\sigma = \bigoplus_{i=0}^{p-1} V_{\lambda^i \chi_g}, \quad V_\tau = \bigoplus_{i=0}^{p-1} V_{\lambda^i \psi_h} \quad (6.7)$$

as V^G -modules.

Suppose that $\phi : V_{\chi_g} \rightarrow V_{\psi_h}$ is a V^G -isomorphism. Choose $0 \neq w \in V_{\chi_g}$ and consider the V^H -submodule of $V_\sigma \oplus V_\tau$ generated by $(w, \phi(w))$. We have $V^H = \bigoplus_{i=0}^{p-1} V^{\lambda^i}$ where V^{λ^i} is the subspace of V transforming according to the character λ^i , and if $u \in V^{\lambda^i}$ then any component operator u_n of u satisfies $u_n w \in V_{\lambda^i \chi_g}$ and $u_n \phi(w) \in V_{\lambda^i \psi_h}$.

Now the argument used in proof of Theorem 5.1 of [DM2] shows first that $V_\sigma \simeq V_\tau$ as V^H -modules, so that $g = h$ by induction; and then that $\chi_g = \psi_h$, as required.

This reduces us to the case that $G = \langle g, h \rangle$ is abelian.

Case B6: $\langle g \rangle \neq G$.

In this case let $g \in H \triangleleft G$ with $[G : H]$ a prime p . We still have the decompositions (6.7), but as far as V^H is concerned, V_τ is an \bar{h} -twisted V^H -module, where \bar{h} is the

automorphism of V^H induced by \bar{h} . But in any case we can still carry out the argument of the last case, leading to the conclusion that there is a proper subspace of $V_\sigma \oplus V_\tau$ invariant under all u_n for $u \in V^H$ and $n \in \mathbb{Z}$ and containing $(w, \phi(w))$. This is impossible.

Case B7: $\langle g \rangle = G$.

In this case, since G is cyclic then the 2-cocycles α_h and α_h may be taken to be trivial. Now the proof of Theorem 5.1 of [DM2] completes the argument.

This finally completes the proof of Theorem 6.1.

7 Proof of Theorem 2

In [DM2] the authors suggested that there should be a Galois correspondence for finite groups of automorphisms of simple VOAs and established such a result for abelian and dihedral group. In lieu of a deeper understanding of this phenomenon, we will establish in this section that such a correspondence holds for nilpotent groups. Namely we will prove Theorem 2.

For each integer n there is a linear map

$$\mu_n : V \otimes V \rightarrow V$$

defined by $\mu_n(v \otimes w) = v_n w$ for $v, w \in V$ and with v_n the n th component operator of v in $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$. Moreover the maps μ_n are G -invariant whenever G is a group of automorphism of V , where as usual G acts on $V \otimes V$ via $g(v \otimes w) = gv \otimes gw$.

Lemma 7.1 *Suppose that $v, w \in V$ and $v \otimes w$ is not G -invariant. Then there is n such that $v_n w$ is not G -invariant.*

Proof: Assume false. If we let $K_n = \ker \mu_n$ then we get $v \otimes w \in K_n + (V \otimes V)^G$ for each n , so that

$$v \otimes w \in (V \otimes V)^G + \bigcap_{n \in \mathbb{Z}} K_n.$$

But Lemma 3.1 of [DM2] tells us that $\bigcap K_n = 0$, whence in fact $v \otimes w \in (V \otimes V)^G$. This contradiction proves the lemma. \square

We turn to the proof of Theorem 2, which we prove by induction on the order of G . Let W be a sub VOA satisfying $V^G \subset W \subset V$. By Proposition 3.3 of [DM2] it suffices to show that $W = V^H$ for some subgroup H of G . As in [DM2], V decomposes according to the simple G -modules, $V = \bigoplus_{\chi \in \text{Irr}(G)} V^\chi$ and $W = \bigoplus_{\chi \in \text{Irr}(G)} (W \cap V^\chi)$.

Now if $W = V^G$ there is nothing to prove, so we may assume that $W \cap V^\chi \neq 0$ for some $\chi \in \text{Irr}(G)$ with $\chi \neq 1_G$. Pick some $0 \neq w \in W \cap V^\chi$. As G is nilpotent we may choose a non-identity g in the center of G . Then g acts on $W \cap V^\chi$ as multiplication by a scalar, say λ . By consideration of $Y(w, z)w$ we see via Lemma 3.1 of [DM2] that also W contains nonzero vectors on which g acts as multiplication by λ^2 , and similarly W contains nonzero vectors on which g acts as multiplication by any power of λ .

We are going to show that we may assume the existence of a non-identity element h in the center of G such that if $Z = \langle h \rangle$ then $W^Z \neq V^G$. If the scalar λ in the preceding paragraph is equal to 1 we may take $g = h$, since in this case $W^Z \supset W \cap V^\chi \neq 0$. If $\lambda \neq 1$ choose, as we may, some $0 \neq x \in W$ such that $gx = \lambda^{-1}x$. Consideration of $Y(w, z)x$ shows that g fixes each $w_n x$, so we may as well assume that each $w_n x$ lies in V^G . Now Lemma 7.1 shows that $w \otimes x \in (V \otimes V)^G$. This can only happen when χ is a linear character (i.e., of degree 1), in which case we may choose h to lie in the kernel of χ if this latter group has order bigger than 1. If it does not then G is a cyclic group, in which case the theorem has already been established in [DM2].

So indeed we may assume that $W^Z \neq V^G$ for some $1 \neq Z = \langle h \rangle < Z(G)$.

Now consider the VOA V^Z . It is simple by Theorem 4.4 of [DM2] and admits G/Z as a faithful automorphism group by Proposition 3.3 of [DM2]. Since W^Z is a sub VOA of V^Z which contains $V^G = (V^Z)^{G/Z}$ then by induction $W^Z = (V^Z)^{H/Z}$ for some subgroup $H/Z < G/Z$. That is, $W^Z = V^H$. Note that since $W^Z \neq V^G$ then $H \neq G$ by Proposition 3.3 (loc. cit.).

So now we are in the situation that W is a sub VOA of V which contains V^H for some proper subgroup $H < G$. By induction, $W = V^K$ for some subgroup $K < H$, and the theorem is proved. \square

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